

Problem Set #4: Numerical Optimal Control (cont'd)

Everybody knows how to swing from playgrounds. Our intuition helps us to swing higher and higher. But what is the Physics behind it? The first task of the problem set is to understand how a swing can be described mathematically. You then explore how swinging can be posed as a control problem. And, of course, you are asked to design a controller that really makes you swing (in the continuation of this problem set).

Hint: In order to familiarize yourself with solution techniques for optimal control problems, please check the examples that are provided with CasADi [3].

Considered Model

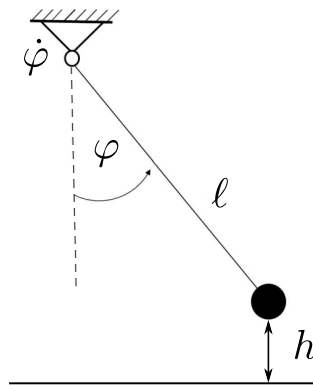


Figure 1: Schematic of a swing.

A swing, see Figure 1, can be modeled by means of an Ordinary Differential Equation (ODE)

$$\frac{d}{dt}(m\ell^2\dot{\varphi}) = -mg\ell\sin(\varphi) - c\ell^2\dot{\varphi}, \quad \varphi(0) = \varphi_0, \quad \dot{\varphi}(0) = \dot{\varphi}_0, \quad \ell(0) = \ell_0, \quad (1)$$

with parameters $m = 10$, $g = 9.81$, $c = 1.5 > 0$, cf. [2] and [1]. The units of the angle φ and the length ℓ are radians and meter, respectively.

As a preparatory step, rewrite the model in state-space form such that

$$x = \begin{bmatrix} \phi \\ \dot{\phi} \\ \ell \end{bmatrix} \quad \text{and} \quad u = \dot{\ell}.$$

Exercise 1: Open-loop Optimal Control of a Swing

- 1) Formulate an Optimal Control Problem (OCP) such that a person swings from an initial position x_0 to a terminal position $x(T) = x_T$ with $0 < T < \infty$. Discuss the necessity of input/state constraints. What behavior do you expect from the simulations? Solve one of the formulated OCPs numerically using CasADi [3] and Matlab. Interpret your results. Are they physically meaningful?

Hint: Use the Casadi.opti object.

- 2) Formulate an OCP such that a person maximizes its terminal height $h(T)$ relative to the ground, see Figure 1. How can the height be expressed in terms of the state variables? What is the physical interpretation of the thus-formulated OCP? Discuss the necessity of input/state constraints. Solve the formulated OCP numerically.
- 3) Adjust your problem formulation such that a person maximizes its terminal height $h(T)$ relative to the ground whilst minimizing the use of the input. Solve the formulated OCP numerically and compare it to your previous result.
- 4) Formulate an OCP such that a person tracks the periodic reference angle $\alpha(t) = \frac{23\pi}{180} \cos\left(\sqrt{\frac{g}{1.5}} t + 0.2\right)$ for all $t \in [0, T]$. Solve the formulated OCP numerically. How do your tuning parameters affect the performance of the open-loop simulations? Plot also the energies (kinetic, potential, dissipation) of the system, and interpret them.

References

- [1] Gabriel, P: *Physikalische Analyse von Schwingbewegungen im Alltag* (Hausarbeit). 2005.
- [2] Magnus, M; Popp, K; Sexto, W: *Schwingungen. Physikalische Grundlagen und mathematische Behandlung von Schwingungen*. 9. Auflage. Springer Vieweg, Wiesbaden, 2013.
- [3] Andersson, J.; Åkesson, J. & Diehl, M. *CasADi: A symbolic package for automatic differentiation and optimal control* Recent advances in algorithmic differentiation, Springer, 2012, 297-307.